
Turbulence and CFD models: Theory and applications

Roadmap

- 1. Recap of Reynolds decomposition and time averaging**
- 2. Data analysis and statistical tools used turbulence modeling**

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- 1. Recap of Reynolds decomposition and time averaging**
2. Data analysis and statistical tools used turbulence modeling

Recap of Reynolds decomposition and time averaging

- Let us recall the Reynolds decomposition for a given field ϕ ,

The diagram illustrates the Reynolds decomposition. At the top, a grey box labeled "Instantaneous value" has a downward arrow pointing to the equation $\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}) + \phi'(\mathbf{x}, t)$. Below the equation, a grey box labeled "Mean part" has an upward arrow pointing to the $\bar{\phi}(\mathbf{x})$ term. To the right, a grey box labeled "Fluctuating part" has an upward arrow pointing to the $\phi'(\mathbf{x}, t)$ term.

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}) + \phi'(\mathbf{x}, t)$$

- It basically states that the instantaneous value of the field ϕ , can be decomposed in two parts, one mean part and one fluctuating part.
- In our notation, the overbar represents the mean part, and the prime (or apostrophe) represents the fluctuating part.
- The terms mean part and fluctuating part, sometimes are referred to as average part and perturbation part, respectively.
- Both terminologies can be used with no problems.

Recap of Reynolds decomposition and time averaging

- Notice that in this decomposition, the mean part does not fluctuate in time.

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}) + \phi'(\mathbf{x}, t)$$



Stationary mean part

- We are dealing with stationary turbulence.
- However, it might happen that the mean part exhibits long period oscillations that are not turbulent in nature.
- When this is the case, we are talking about nonstationary turbulence,

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}, t) + \phi'(\mathbf{x}, t)$$



Nonstationary mean part

- The previous definition is general and it can be used with stationary turbulence.

Recap of Reynolds decomposition and time averaging

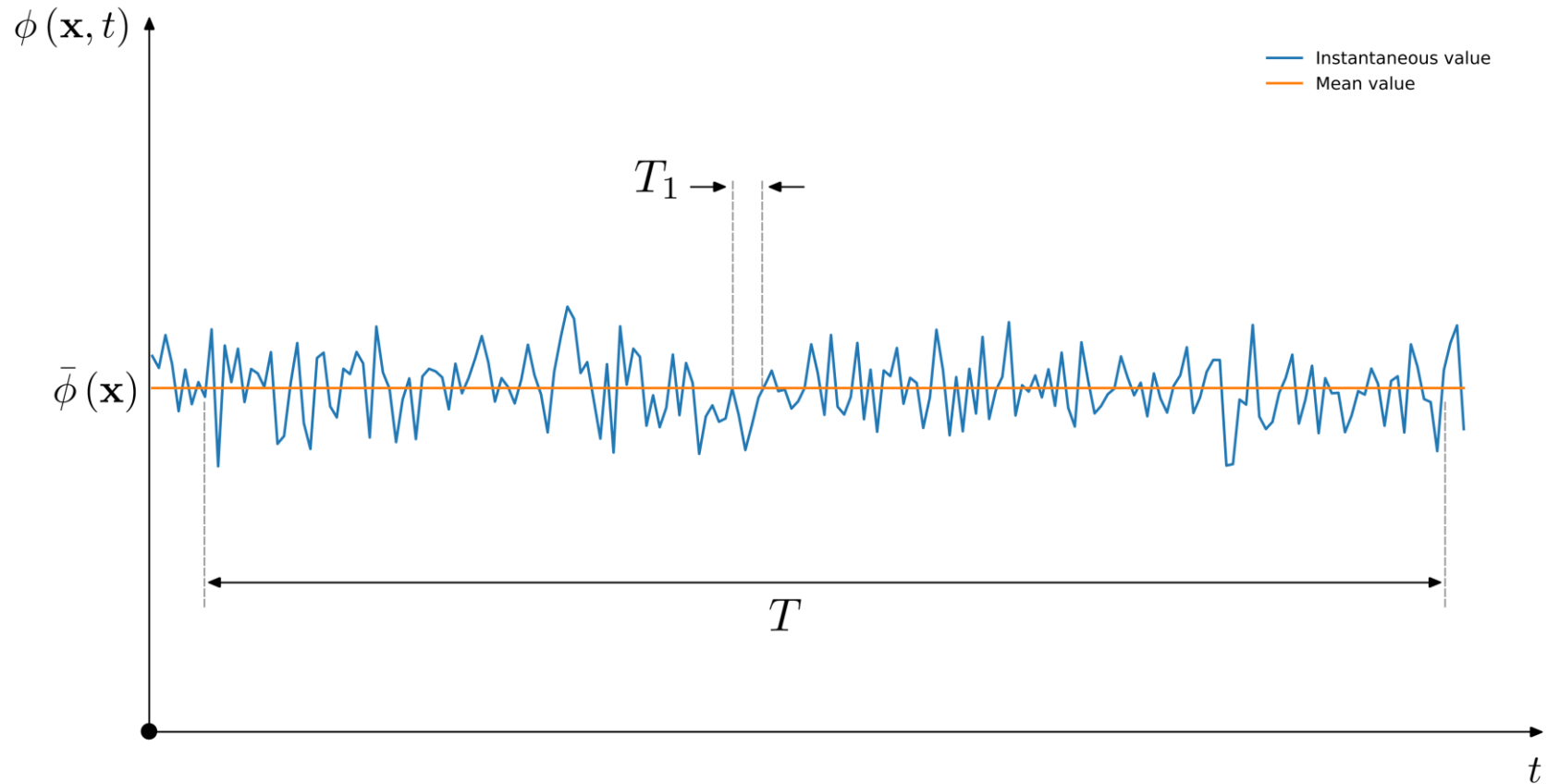
- To compute the mean (or average), we can use time averaging,

$$\bar{\phi}(\mathbf{x}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \phi(\mathbf{x}, t) dt$$

- Here, T represents the averaging interval.
- This interval must be large compared to the typical time scales of the fluctuations so it will yield to a stationary state.
- Time averaging is appropriate for stationary turbulence or slowly varying turbulent flows, *i.e.*, a turbulent flow that, on average, does not vary much with time.
- Notice that we are not making the distinction between steady or unsteady flows.
- The time average can be in iterative or in time.
- In statistically steady turbulence, if we take the average between different ranges or values of t , we will get approximately the same mean value.

Recap of Reynolds decomposition and time averaging

- Time averaging for stationary turbulence.
- The averaging interval T must be large compared to the typical time scales of the fluctuations T_1 so it will yield to a stationary state.



Recap of Reynolds decomposition and time averaging

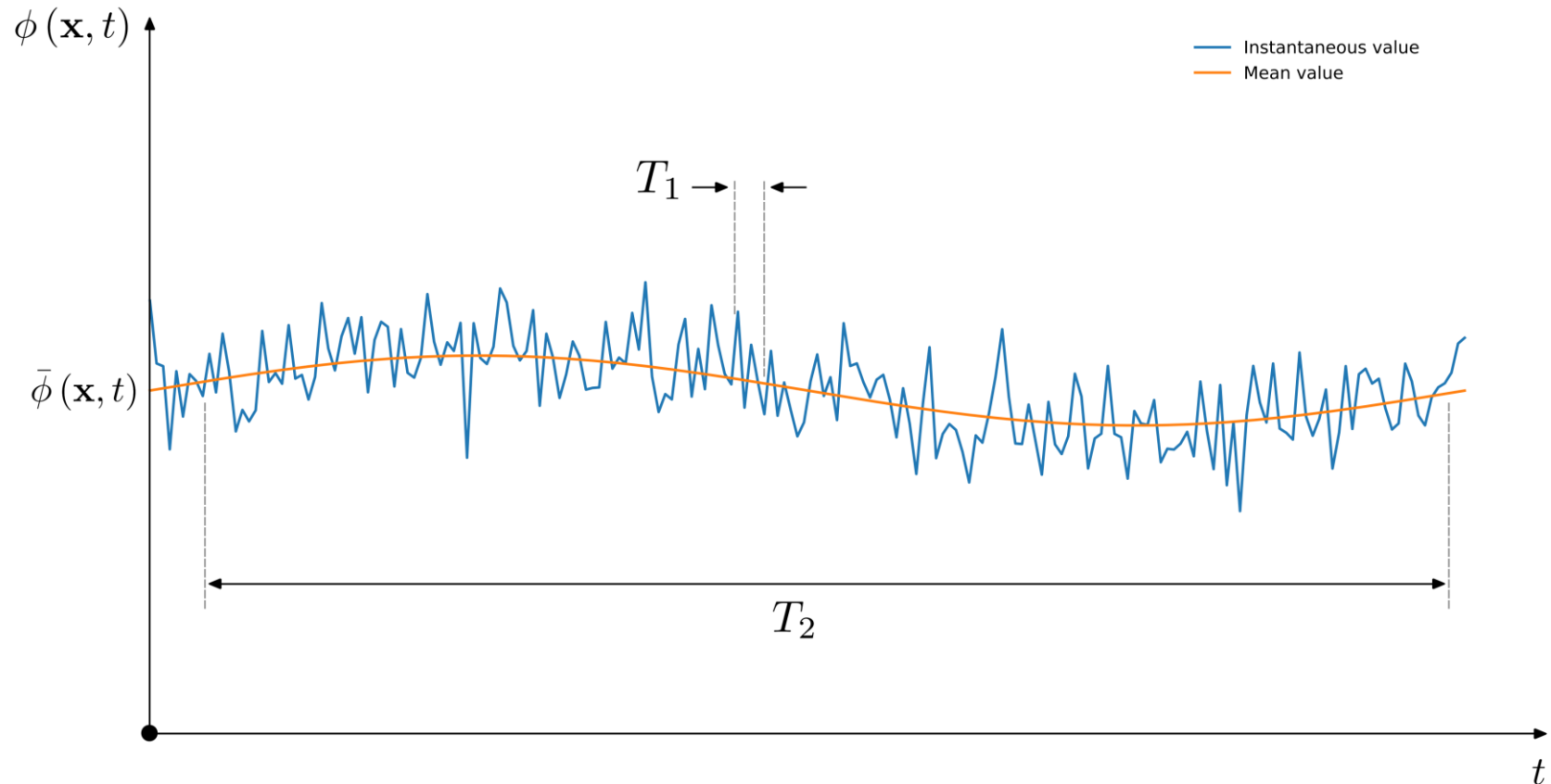
- If the mean quantity varies in time, we can compute the mean (or average), as follows,

$$\bar{\phi}(\mathbf{x}, t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \phi(\mathbf{x}, t) dt \quad T_1 \ll T \ll T_2$$

- Where T_2 is the time scale characteristic of the slow variations in the flow that we do not wish to regard as belonging to the turbulence.
- The time scales T_1 and T_2 differ by several orders of magnitude.
- In this situation we are dealing with nonstationary (unsteady) turbulence. Therefore, we need to use unsteady solvers.
- However, in this kind of situations it might be better to use ensemble averaging.

Recap of Reynolds decomposition and time averaging

- Time averaging for nonstationary turbulence.
- In this case, T_2 is the time scale characteristic of the slow variations in the flow that we do not wish to regard as belonging to the turbulence.
- The time scales T_1 and T_2 differ by several orders of magnitude.




Recap of Reynolds decomposition and time averaging

- Ensemble averaging is appropriate for unsteady turbulence or any kind of flow.
- In this case, the mean value is computed as follows,

$$\bar{\phi}(\mathbf{x}, t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi(\mathbf{x}, t)$$

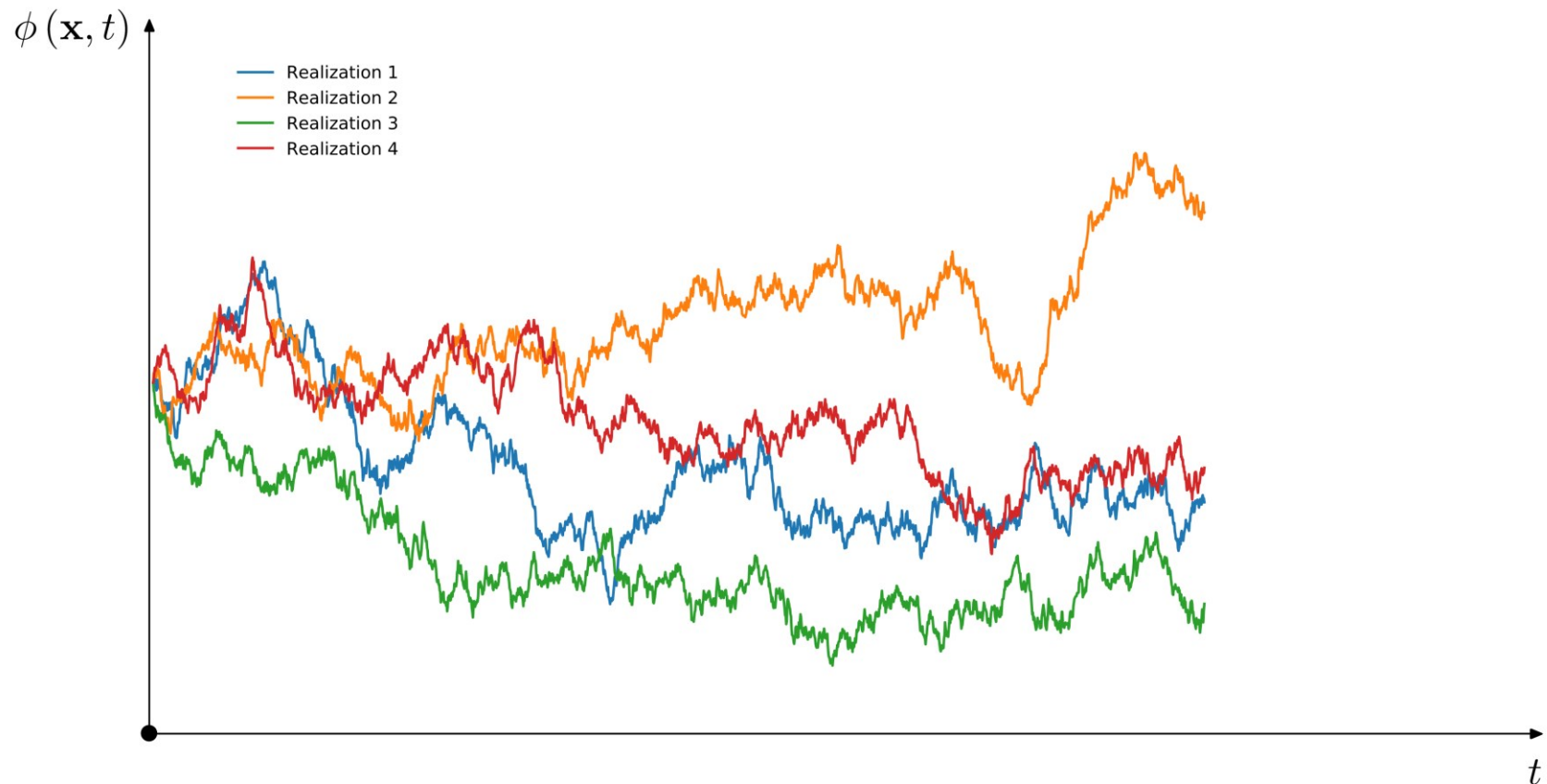
Number of realizations



- In ensemble averaging, the number of realizations (or experiments) must be large enough to eliminate the effects of fluctuations. This type of averaging can be used with steady or unsteady flows.
- In this kind of situations, it might be better to use ensemble averaging.
- However, ensemble averaging requires running many experiments.
- This approach is better fit for experiments as CFD is more deterministic.
- Ensemble average can also be used when having periodic or repetitive signal behavior.
- However, you will need to run for long times in order to take good averages.

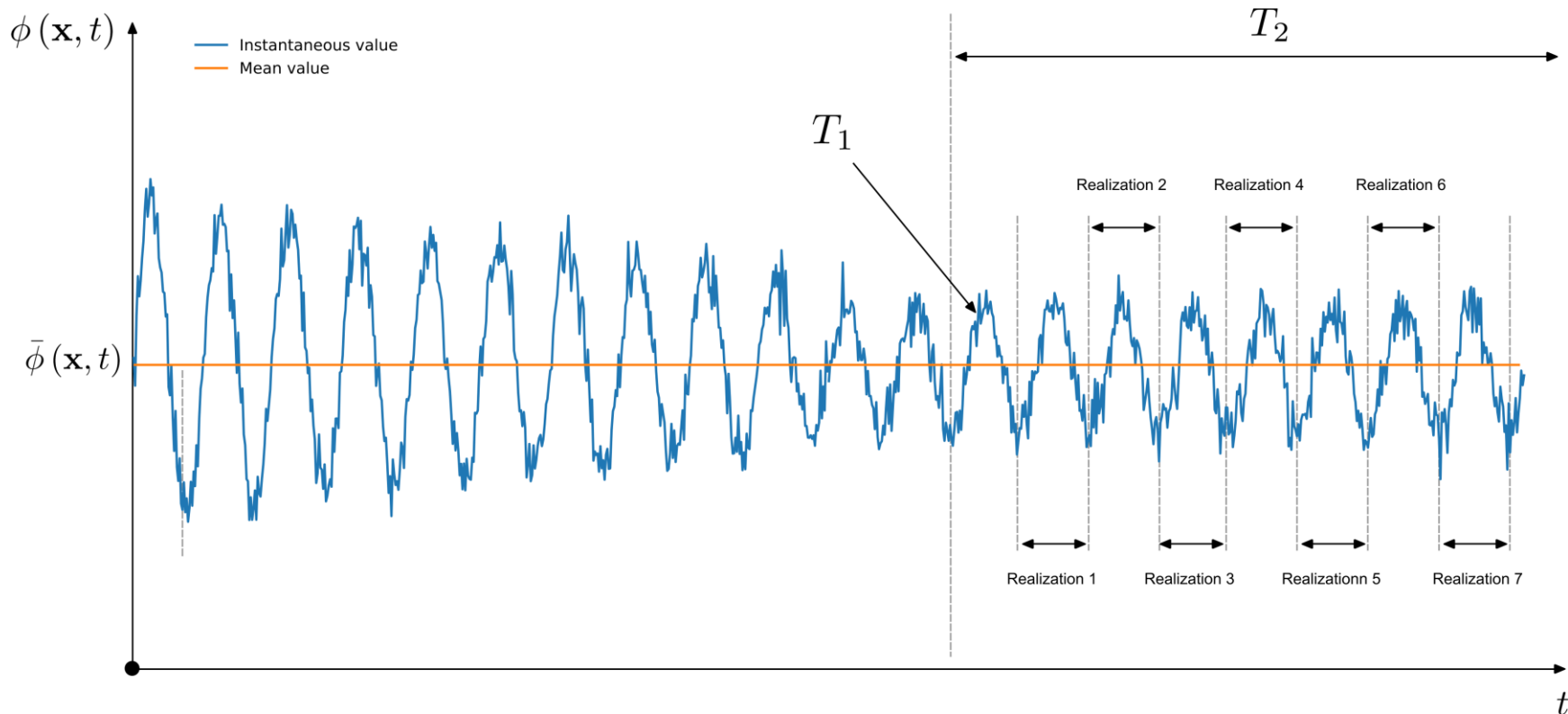
Recap of Reynolds decomposition and time averaging

- Ensemble averaging can be used for stationary and nonstationary turbulence.
- In this case, the final average is the summation of the averages of each realization.
- The number of realizations (or experiments), must be large enough to eliminate the effects of fluctuations.



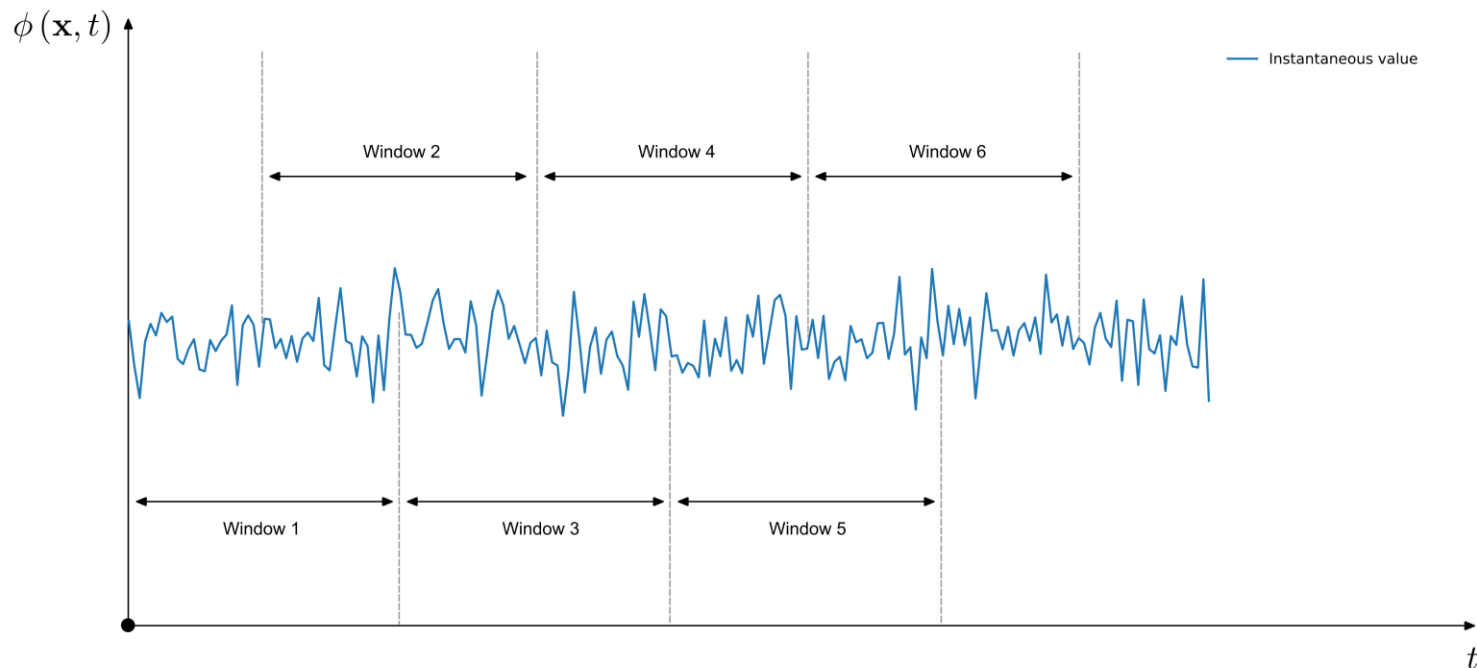
Recap of Reynolds decomposition and time averaging

- Ensemble averaging can be used for stationary and nonstationary turbulence.
- In this case, the ensemble average can be seen as the summation of many similar periods (or realizations).
- T_2 must be large enough to eliminate the effects of fluctuations.
- The time scales T_1 and T_2 differ by several orders of magnitude.



Recap of Reynolds decomposition and time averaging

- Time ensembles must be statistically stationary for the moments to have a significant meaning (mean, variance, skewness, kurtosis).
- A statistically stationary flow is defined as a flow in which statistical parameters do not depend on the interval in time used to evaluate them.
- To determine if the flow is statistically stationary, one can use the overlapped windowing techniques.
- However, this technique is effective if at least two of the windows are completely independent.
- The number of windows and the number of samples in each window is generally determined by the number of samples available in the signal.



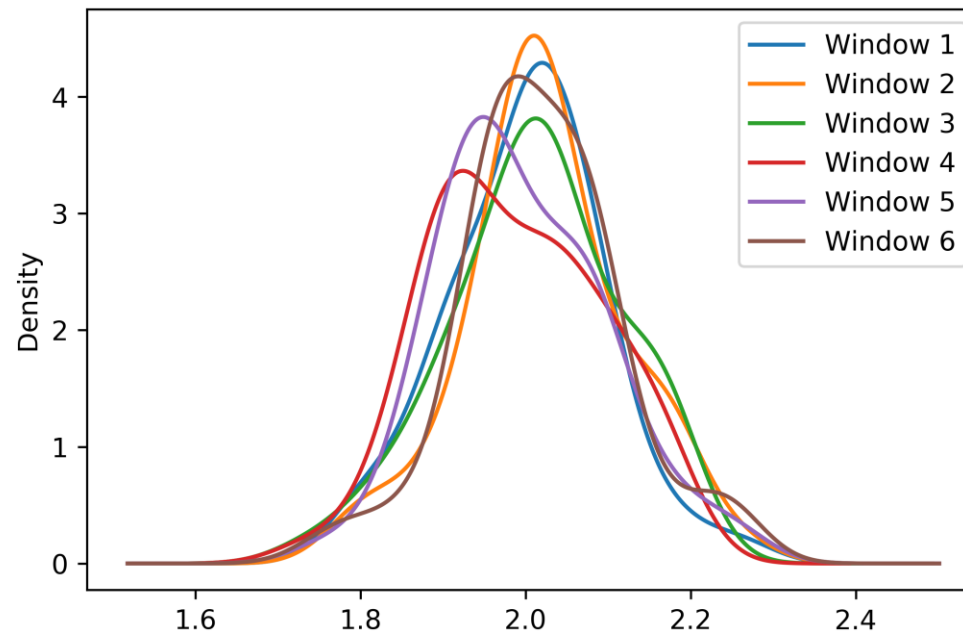
Recap of Reynolds decomposition and time averaging

- After constructing the windows, one can compute the first four statistical moments, that is, mean, variance skewness, and kurtosis.
- The higher moments, namely, variance, skewness and kurtosis, are defined relative to the mean.
- The variance provide information on the spread of the data away from the mean.
- Skewness gives the amount of time the signal is above or below the mean. That is, the deviation of the data from the mean value.
- Kurtosis gives the amount of time the signal is away from the mean or flatness of the curve.
- Symmetric signals have a skewness of zero.
- Additionally, one can compute the standard deviation (std) which the square root of the variance.

	Mean	Variance	Skewness	Kurtosis	Std
Window 1	2.0012	0.0086	0.0162	0.3058	0.0932
Window 2	2.0229	0.0093	-0.7422	0.3434	0.0964
Window 3	2.0112	0.0104	-0.2879	-0.2106	0.1023
Window 4	1.9870	0.0105	0.0702	-0.7082	0.1027
Window 5	1.9953	0.0099	0.4067	0.0302	0.0999
Window 6	2.0175	0.0091	0.1039	0.7774	0.0958

Recap of Reynolds decomposition and time averaging

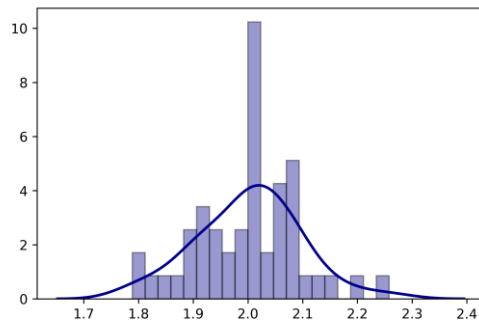
- We can also plot the density plot.
- A density plot is a smoothed, continuous version of a histogram estimated from the data.
- The most common form of estimation is known as kernel density estimation.



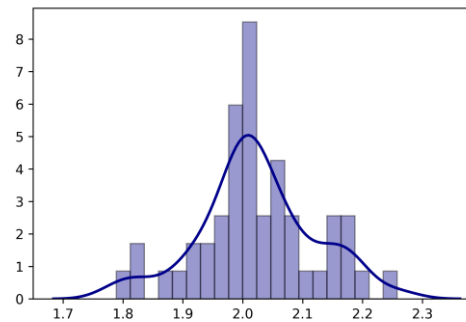
Recap of Reynolds decomposition and time averaging

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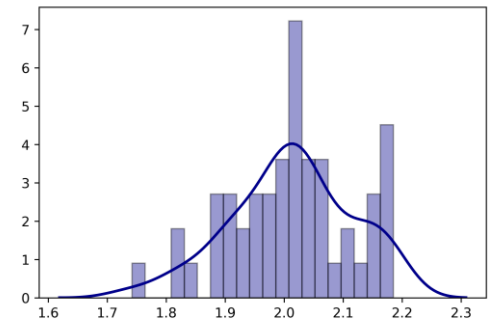
Window 1



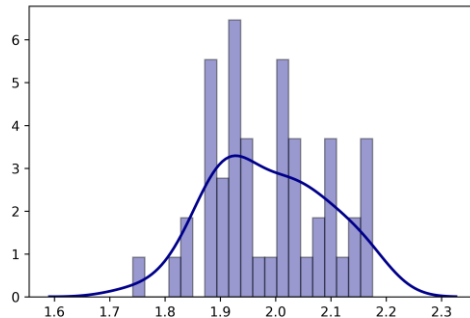
Window 2



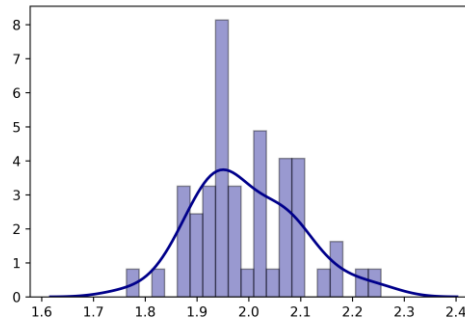
Window 3



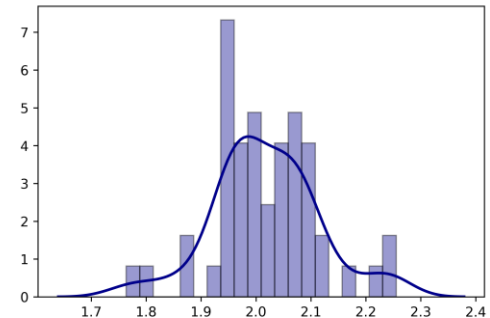
Window 4



Window 5

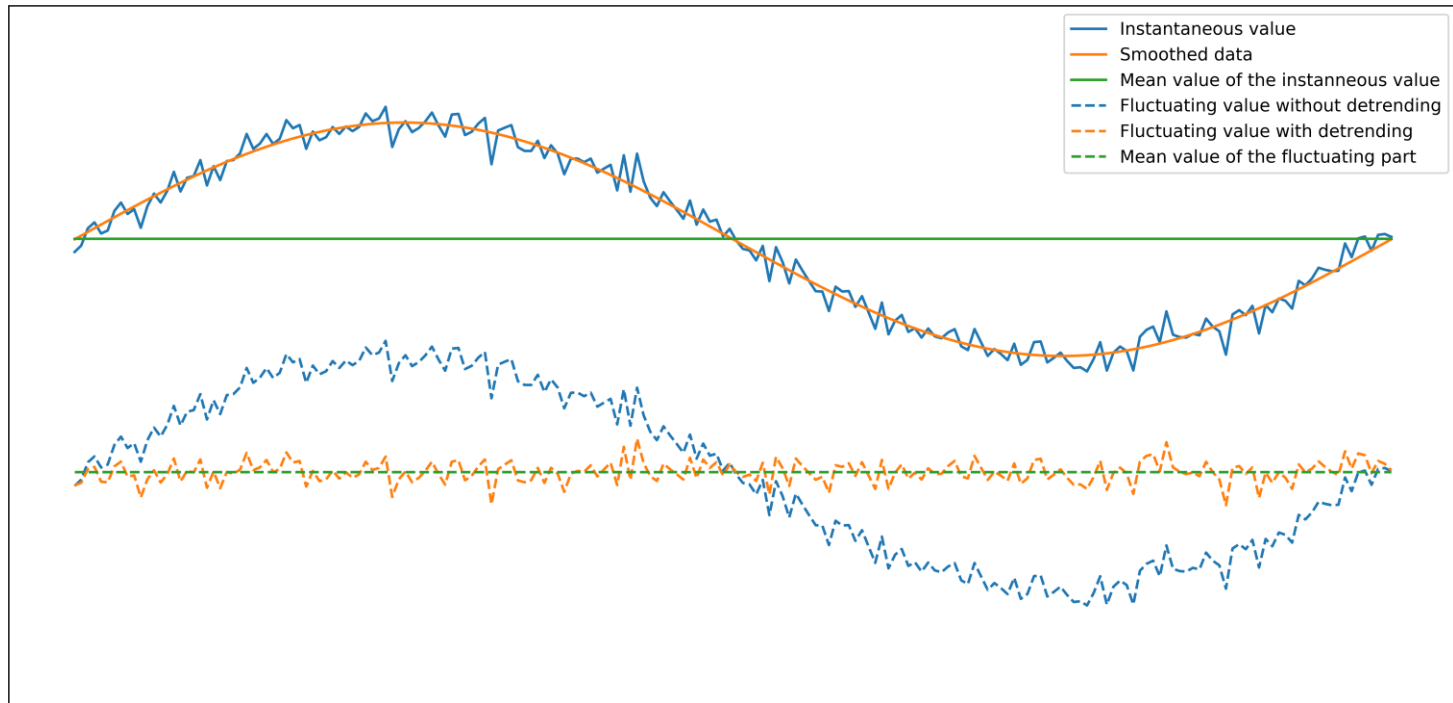


Window 6



Recap of Reynolds decomposition and time averaging

- In the case of nonstationary turbulence, remember to detrend your data in order to compute the TKE.
- If you do not detrend the data, the fluctuating part will exhibit the same slow variations in the flow (the variations that we do not wish to regard as belonging to the turbulence) as the instantaneous data.
- These situations usually happen with seasonal data.
- There are many methods to detrend data, the easiest one in my opinion is rolling averages.
- Look at signal processing or financial methods for more information.



Recap of Reynolds decomposition and time averaging

- Any of the previous time averaging rules can be used without loss of generality.
- Before continuing, let us recall a few averaging rules.

$$\bar{\phi}' = 0,$$

$$\bar{\bar{\phi}} = \bar{\phi},$$

$$\bar{\bar{\phi}} = \overline{\bar{\phi} + \phi'} = \bar{\phi},$$

$$\overline{\phi + \varphi} = \bar{\phi} + \bar{\varphi},$$

$$\overline{\bar{\phi}\varphi} = \bar{\bar{\phi}}\bar{\varphi} = \bar{\phi}\bar{\varphi},$$

$$\overline{\bar{\phi}\varphi'} = \bar{\phi}\bar{\varphi}' = 0,$$

$$\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \bar{\phi}}{\partial x},$$

$$\overline{\phi\varphi} = \overline{(\bar{\phi} + \phi')(\bar{\varphi} + \varphi')}$$

$$= \overline{\bar{\phi}\bar{\varphi} + \bar{\phi}\varphi' + \bar{\varphi}\phi' + \phi'\varphi'}$$

$$= \overline{\bar{\phi}\bar{\varphi}} + \overline{\bar{\phi}\varphi'} + \overline{\bar{\varphi}\phi'} + \overline{\phi'\varphi'}$$

$$= \bar{\phi}\bar{\varphi} + \overline{\phi'\varphi'},$$

$$\overline{\phi'^2} \neq 0,$$

$$\overline{\phi'\varphi'} \neq 0,$$

$$\overline{\int \phi ds} = \int \bar{\phi} ds$$

Recap of Reynolds decomposition and time averaging

- Let us demonstrate the following two averaging rules, $\overline{\overline{\phi}} = \overline{\phi}$ and $\overline{\phi'} = 0$.
- The average of the average is equal to the average,

$$\overline{\overline{\phi}}(\mathbf{x}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \overline{\phi}(\mathbf{x}) dt = \overline{\phi}(\mathbf{x})$$

- The average of the fluctuating part is equal zero,

$$\overline{\phi'}(\mathbf{x}, t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \underbrace{[\phi(\mathbf{x}, t) - \overline{\phi}(\mathbf{x})]}_{\substack{\uparrow \\ \phi'(\mathbf{x}, t) = \phi(\mathbf{x}, t) - \overline{\phi}(\mathbf{x})}} dt = \overline{\phi}(\mathbf{x}) - \overline{\overline{\phi}}(\mathbf{x}) = 0$$

Recap of Reynolds decomposition and time averaging

- The time average of an unsteady term is zero for stationary turbulence,

$$\frac{\partial \phi}{\partial t} = 0 \quad \longleftarrow \quad \text{In stationary turbulence}$$

- In nonstationary turbulence, the unsteady term can be decomposed as follows,

$$\begin{aligned} \frac{1}{T} \int_t^{t+T} \frac{\partial}{\partial t} (\bar{\phi} + \phi') dt &= \frac{\bar{\phi}(\mathbf{x}, t+T) - \bar{\phi}(\mathbf{x}, t)}{T} + \dots \\ &\dots + \frac{\phi'(\mathbf{x}, t+T) - \phi'(\mathbf{x}, t)}{T} \end{aligned}$$

- The second term can be neglected provided $|\phi'| \ll |\bar{\phi}|$. This is one of those questionable approximation in turbulence modeling.
- Also, as we are assuming that T is very small relative to the time scale of the mean flow, we obtain,

$$\overline{\frac{\partial \phi}{\partial t}} \approx \frac{\partial \bar{\phi}}{\partial t}$$

Recap of Reynolds decomposition and time averaging

- So far, we have considered averages of linear quantities.
- When we average the product of two properties, say ϕ and φ , we obtain,

$$\begin{aligned}\overline{\phi\varphi} &= \overline{(\bar{\phi} + \phi')(\bar{\varphi} + \varphi')} \\ &= \overline{\bar{\phi}\bar{\varphi} + \bar{\phi}\varphi' + \bar{\varphi}\phi' + \phi'\varphi'} \\ &= \overline{\bar{\phi}\bar{\varphi}} + \overline{\bar{\phi}\varphi'} + \overline{\bar{\varphi}\phi'} + \overline{\phi'\varphi'} \\ &= \bar{\phi}\bar{\varphi} + \overline{\phi'\varphi'}\end{aligned}$$

- From this relation, there is no reason for the mean of the product of two fluctuating quantities to vanish.
- Also, the mean value of the product of two quantities, $\overline{\phi\varphi}$, is not necessarily equal to the product of the mean values, $\bar{\phi}\bar{\varphi}$.
- Finally, the product of the fluctuations are said to be correlated (or uncorrelated) if,

$$\overline{\phi'\varphi'} \neq 0$$

Correlated

$$\overline{\phi'\varphi'} = 0$$

Uncorrelated

Recap of Reynolds decomposition and time averaging

- Similarly, for the triple product, we find,

$$\overline{abc} = \bar{a}\bar{b}\bar{c} + \overline{a'b'\bar{c}} + \overline{b'c'\bar{a}} + \overline{a'c'\bar{b}} + \overline{a'b'c'}$$

- All the fluctuating terms, namely, a', b', c' , have zero mean.
- And as for the quadratic terms, there is no reason for the mean of the product of the three fluctuating quantities $\overline{a'b'c'}$ to vanish.
- Also, the following averaging rules holds,

$$\overline{\bar{a}\bar{b}c'} = 0$$

$$\overline{\bar{a}b'c'} \neq 0$$

- We can similarly develop relations for the quadruple product, and so on.

Recap of Reynolds decomposition and time averaging

- Before moving to the next section, look at the circled term.

$$\overline{\phi\psi} = \bar{\phi}\bar{\psi} + \overline{\phi'\psi'}$$

- What is this term?
- Where do you find a similar product in the governing equations?

- If we solve for $\overline{\phi'\psi'}$, we obtain,

$$\overline{\phi'\psi'} = \overline{\phi\psi} - \bar{\phi}\bar{\psi}$$

- And this suggest that in order to compute $\overline{\phi'\psi'}$, we need to compute $\overline{\phi\psi}$ and $\bar{\phi}\bar{\psi}$.

Recap of Reynolds decomposition and time averaging

- The term $\overline{\phi'\varphi'}$ is equivalent to the Reynold stresses τ^R .

$$\tau^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = - \begin{pmatrix} \overline{\rho u' u'} & \overline{\rho u' v'} & \overline{\rho u' w'} \\ \overline{\rho v' u'} & \overline{\rho v' v'} & \overline{\rho v' w'} \\ \overline{\rho w' u'} & \overline{\rho w' v'} & \overline{\rho w' w'} \end{pmatrix}$$

- We now have a way to compute the Reynolds stresses from the resolved field (in DNS, LES, or from experimental data),

$$\overline{\mathbf{u}'\mathbf{u}'} = \overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\bar{\mathbf{u}}$$

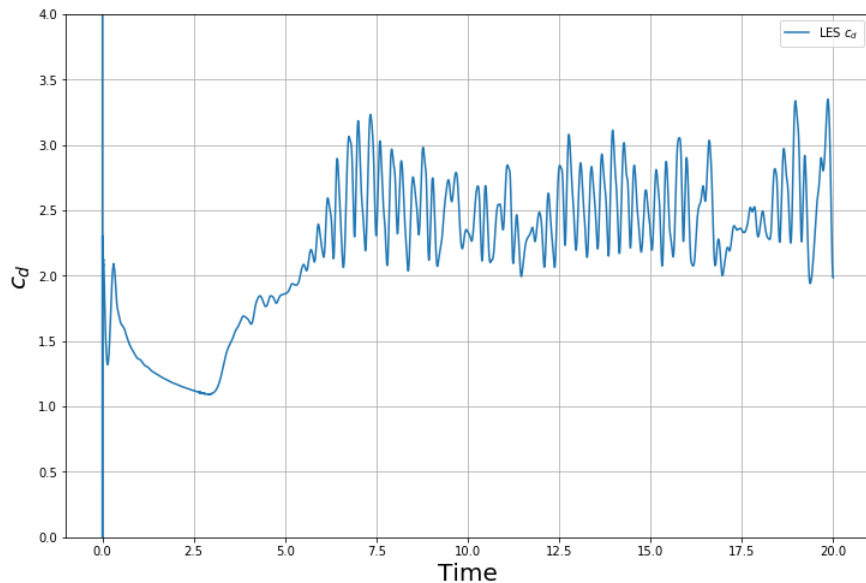
- Remember, you should compute the two quantities in the right-hand side.
- Namely, the average of the product of the velocities $\overline{\mathbf{u}\mathbf{u}}$, and the product of the averaged velocities $\bar{\mathbf{u}}\bar{\mathbf{u}}$.
- And these averaged quantities are computed from unsteady data.

Roadmap

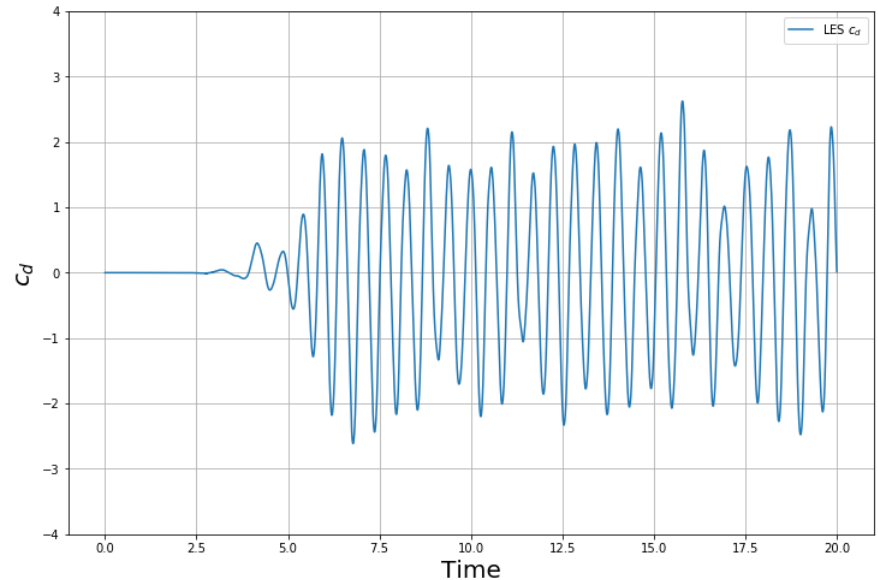
- 1. Recap of Reynolds decomposition and time averaging**
- 2. Data analysis and statistical tools used turbulence modeling**

Sampling of integral quantities

- It is important to compute integral quantities when running turbulent simulations.
- You can sample the integral quantities using iterative marching or time marching.
- Then, you can compute the descriptive statistics of the signal.
- Many integral quantities can be sampled, such as, mass flow, heat transfer rate, shear stresses, friction coefficient, forces, and so on.



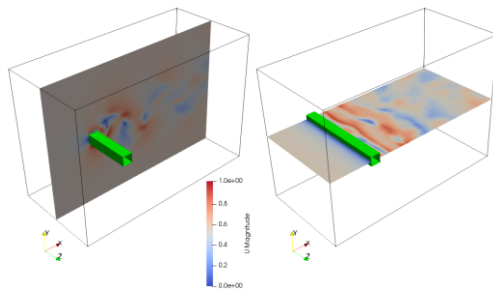
Mean value	2.495
Standard deviation	0.286
Variance	0.0822



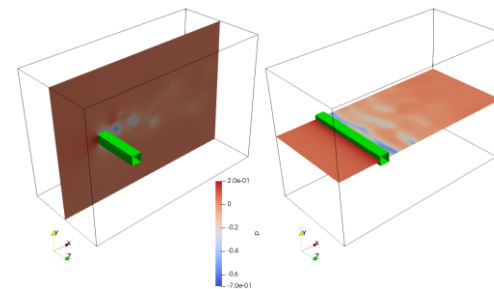
Mean value	-0.010
Standard deviation	1.355
Variance	1.837

Mean values

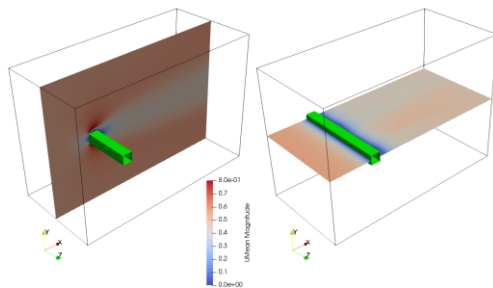
- For most engineering applications, we are interested in computing the mean quantities.
- Therefore, do not forget to compute the mean values of the quantities of interest.
- The mean values can be computed for a signal (time series) or the field variables.
- Sometimes it is very difficult to determine if the signal is statistically stationary, in this situation, we can use the mean field.
- If the mean fields are not changing between iterations, we can say that the solution is statistically stationary.



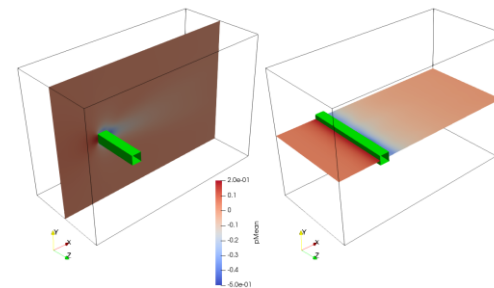
Instantaneous velocity



Instantaneous pressure



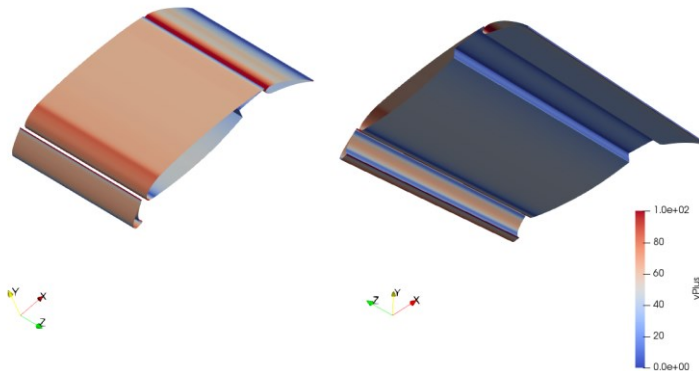
Mean velocity



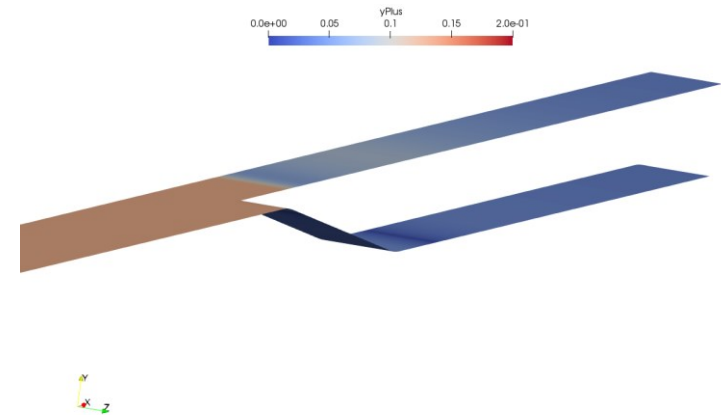
Mean pressure

Minimum, maximum, and averaged values

- Besides the mean values, it is also recommended to compute (and monitor) the minimum and maximum values of the field variables.
- For example, if at any point of the simulation a quantity is oscillating too much or the minimum or maximum value is unrealistic, you might stop the simulation and revise the case setup.
- In the figure below, we show this scenario for y^+ . But you can do it with any quantity, e.g., pressure, velocity, temperature, turbulent kinetic energy, and so on.
- Remember, there are some quantities that are strictly bounded, so it is a good idea to monitor those quantities.



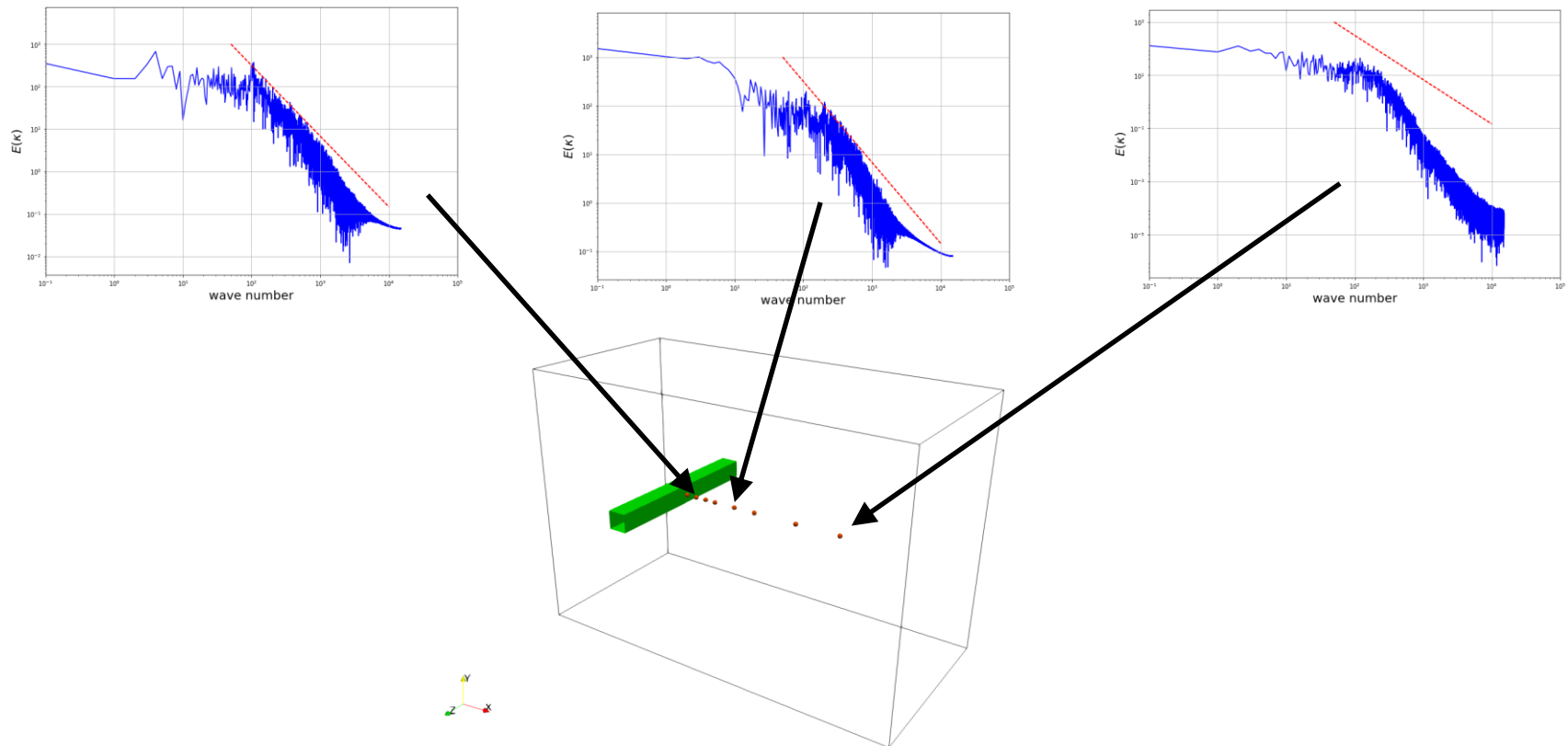
airfoil y^+ : min = 3.3170682, max = 122.32767, average = 42.357341
flap y^+ : min = 9.6251989, max = 447.31831, average = 47.411466
slat y^+ : min = 14.072073, max = 305.59193, average = 93.392662



walls y^+ : min = 0.00135130, max = 0.290177, average = 0.0664195

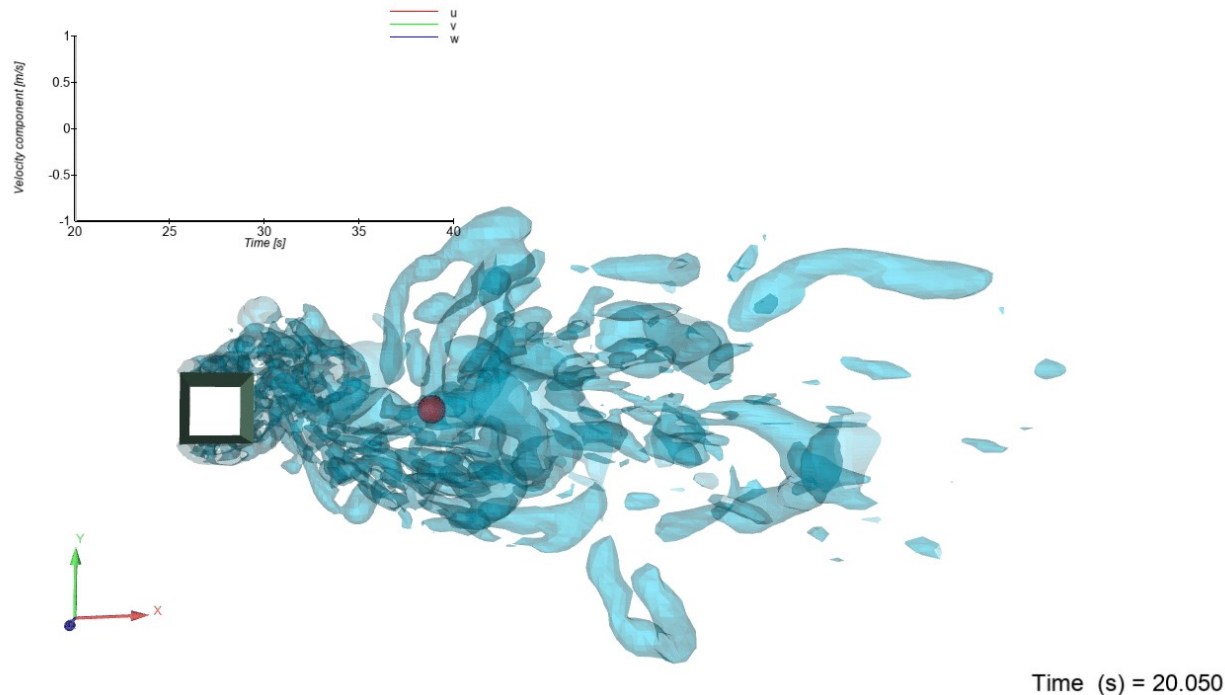
Turbulent power spectrum

- The turbulent power spectrum represents the distribution of the kinetic energy across the various length scales.
- It is a direct indication of how energy is dissipated with eddies size.
- These plots are local and are obtained by sampling in time the kinetic energy in various locations of the domain (a lot of data needs to be gathered).



Turbulent power spectrum

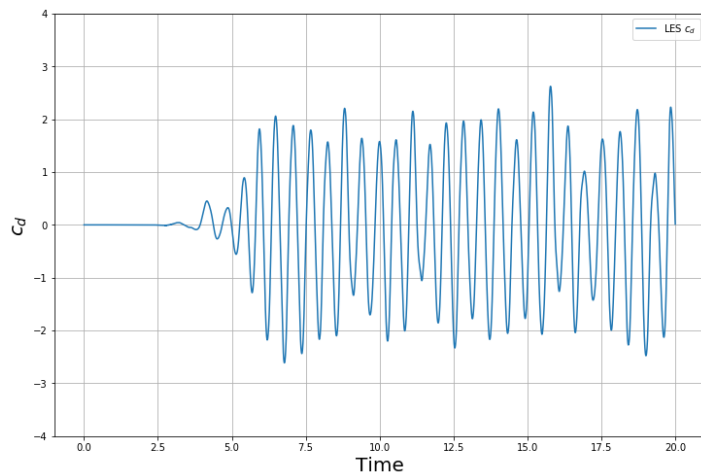
- To plot the energy spectrum, we need to sample the velocity field in a location behind the wake (the red sphere in this case).
- Then, by using signal processing (FFT), we can convert physical space into frequency space.
- Remember to detrend the data in the case of nonstationary turbulence. If you do not this, you will highly overpredict TKE.



Sampling of integral quantities – Dominant frequency

- In many unsteady simulations there is vortex shedding.
- The shedding frequency can be computed from the time signal of a sampled integral quantity (e.g., forces).
- The dominant frequency can be computed using signal processing methods (e.g. periodogram).

Input signal



Signal processing

Power spectral density of the input signal

